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**3 (Sem-4/CBCS) MAT HC 1**

**2022**

**MATHEMATICS**

(Honours)

Paper : MAT-HC-4016

**(Multivariate Calculus)**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer **any ten** : 1×10=10

(i) Find the domain of  $f(x, y) = \frac{1}{\sqrt{x-y}}$ .

(ii) How is directional derivative of a function at a point related to the gradient of the function at that point?

(iii) Define harmonic function?

(iv) Define  $\iint_R f(x, y) dA$ .

Contd.

- (v) Write the value of  $\bar{\nabla}(f^n)$ .
- (vi) Define critical point.
- (vii) Define relative extrema for a function of two variables.
- (viii) When is a curve said to be positively oriented?
- (ix) Describe the fundamental theorem of line integral.
- (x) When is a surface said to be smooth?
- (xi) Compute  $\int_1^4 \int_{-2}^3 \int_2^5 dx dy dz$ .
- (xii) Evaluate  $\lim_{(x,y) \rightarrow (1,3)} \frac{x-y}{x+y}$ .
- (xiii) If  $f(x, y) = x^3y + x^2y^2$ , find  $f_x$ .
- (xiv) When is a line integral said to be path independent?
- (xv) Explain the difference between  $\int_C f ds$  and  $\int_C f dx$ .

2. Answer **any five** questions :  $2 \times 5 = 10$

(a) Sketch the level surface  $f(x, y, z) = c$  if  $(x, y, z) = y^2 + z^2$  for  $c = 1$ .

(b) Determine  $f_x$  and  $f_y$  for  $f(x, y) = xy^2 \ln(x + y)$ .

(c) Find  $\frac{\partial w}{\partial t}$  if  $w = \ln(x + 2y - z^2)$  and  $x = 2t - 1, y = \frac{1}{t}, z = \sqrt{t}$ .

(d) Evaluate  $\int_1^2 \int_0^\pi x \cos y \, dy \, dx$ .

(e) Evaluate  $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 + x - xy - y}{x - y}$ .

(f) Define line integral over a smooth curve.

(g) Find the Jacobian  $\frac{\partial(x, y)}{\partial(u, v)}$  when

$$x = u + 2v, y = 3u - 4v.$$

(h) Using polar coordinates find the limit

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{\tan(x^2 + y^2)}{x^2 + y^2}.$$

3. Answer **any four**: 5×4=20

(a) Describe the graph of the function

$$f(x, y) = 1 - x - \frac{1}{2}y.$$

(b) Use the method of Lagrange's multipliers to find the maximum and minimum values of  $f(x, y) = 1 - x^2 - y^2$  subject to the constraints  $x + y = 1$  with  $x \geq 0, y \geq 0$ .

(c) Evaluate  $\int_C [(y - x)dx + x^2 y dy]$ , where  $C$  is the curve defined by  $y^2 = x^3$  from  $(1, -1)$  to  $(1, 1)$ .

(d) Examine the continuity of the following function at the origin:

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

(e) Find  $\frac{\partial w}{\partial s}$  if  $w = 4x + y^2 + z^3$  where

$$x = e^{rs^2}, \quad y = \ln \frac{r+s}{t} \quad \text{and} \quad z = rst^2.$$

(f) Suppose the function  $f$  is differentiable at the point  $P_0$  and that the gradient at  $P_0$  satisfies  $\Delta f_0 \neq 0$ . Show that  $\Delta f_0$  is orthogonal to the level surface of  $f$  through  $P_0$ .

(g) Compute  $\iint_D \left( \frac{x-y}{x+y} \right)^4 dy dx$  where  $D$  is

the triangular region bounded by the line  $x+y=1$  and the coordinate axes, using change of variables  $u=x-y$ ,  $v=x+y$ .

(h) Find the absolute extrema of  $f(x, y) = 2x^2 - y^2$  on the closed bounded set  $S$ , where  $S$  is the disk  $x^2 + y^2 \leq 1$ .

4. Answer **any four** questions :  $10 \times 4 = 40$

(a) The radius and height of a right circular cone are measured with errors of at most 3% and 2% respectively. Use increments to approximate the maximum possible percentage error in computing the volume of the cone using these measurements and the formula  $V = \frac{1}{3} \pi R^2 H$ .

(b) Let  $f(x, y) = \begin{cases} xy \left( \frac{x^2 - y^2}{x^2 + y^2} \right), & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$

Show that  $f_x(0, y) = -y$  and  $f_x(x, 0) = x$  for all  $x$  and  $y$ . Then show that

$f_{xy}(0, 0) = -1$  and  $f_{yx}(0, 0) = 1$ .

(c) (i) Find the directional derivative of  $f(x, y) = \ln(x^2 + y^3)$  at  $P_0(1, -3)$  in the direction of  $\vec{v} = 2i - 3j$ .

(ii) In what direction is the function defined by  $f(x, y) = xe^{2y-x}$  increasing most rapidly at the point  $P_0(2, 1)$ , and what is the maximum rate of increase? In what direction is  $f$  decreasing most rapidly?

(d) When two resistances  $R_1$  and  $R_2$  are connected in parallel, the total resistance  $R$  satisfies  $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ .

If  $R_1$  is measured as 300 ohms with maximum error of 2% and  $R_2$  is measured as 500 ohms with a maximum error of 3%, what is the maximum percentage error in  $R$ ?

(e) Verify the vector field

$\vec{F} = (e^x \sin y - y)i + (e^x \cos y - x - 2)j$  is conservative. Also find the scalar potential function  $f$  for  $\vec{F}$ .

- (f) (i) Evaluate  $\iiint_D \frac{dx dy dz}{\sqrt{x^2 + y^2 + z^2}}$   
 where  $D$  is the solid sphere  
 $x^2 + y^2 + z^2 \leq 3$ .
- (ii) Find the volume of the solid  $D$ ,  
 where  $D$  is bounded by the  
 paraboloid  $z = 1 - 4(x^2 + y^2)$  the  
 $xy$ -plane.
- (g) (i) Use a polar double integral to show  
 that a sphere of radius  $a$  has  
 volume  $\frac{4}{3}\pi a^3$ .
- (ii) Evaluate  $\int_0^3 \int_0^{\sqrt{9-x^2}} x dy dx$  by  
 converting to polar coordinates.
- (h) State Green's theorem. Verify Green's  
 theorem for the line integral  
 $\oint_C (y^2 dx + x^2 dy)$  where  $C$  is the square  
 having vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$  and  
 $(0, 1)$ .

- (i) State Stokes' theorem. Using Stokes' theorem evaluate the line integral  $\oint_C (x^3 y^2 dx + dy + z^2 dz)$ , where  $C$  is the circle  $x^2 + y^2 = 1$  and in the plane  $z = 1$ , counterclockwise when viewed from the origin.
- (j) A container in  $R^3$  has the shape of the cube given by  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ . A plate is placed in the container in such a way that it occupies that portion of the plane  $x + y + z = 1$  that lies in the cubical container. If the container is heated so that the temperature at each point  $(x, y, z)$  is given by  $T(x, y, z) = 4 - 2x^2 - y^2 - z^2$  in hundreds of degrees Celsius, what are the hottest and coldest points on the plate? You may assume these extreme temperatures exist.